UDK 519.681.2, 519.681.3

Partial Orders in Transition Systems from Resolvable Conflict Event Structures

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Event structures are a well-established model in concurrency theory. Two structurally different methods of associating transition system semantics to event-oriented models are distinguished in the literature. One of them is based on configurations (event sets), the other — on residuals (model fragments). In this paper, we deal with a highly expressive model of event structures — event structures for resolvable conflict (RC-structures) — and provide isomorphism results on these two types of transition systems constructed from RC-structures, in step and partial order semantics.

Keywords: event structures, resolvable conflict, transition systems, partial order, true concurrency

1. Introduction

In [24], Nielsen, Plotkin and Winskel have introduced the concept of prime event structures which are abstract representations of the behaviour of safe Petri nets. This event-oriented model describes a concurrent system by means of a set of events, representing action occurrences, and for every two events it is specified whether one of them is a predecessor for the other (*causality* presented in the form a partial order), whether they exclude each other (*conflict*), or whether they may happen independently (*concurrency*). The behaviour of an event structure is formalized by associating to it a family of configurations, these being sets of events that occur during runs of the represented system. A configuration can also be understood as a state of the represented system, namely the state reached after executing all events in the configuration. Since then, many studies have been conducted on possible relations among events, giving rise to a number of different definitions of event structures. Flow event structures [9] drop the requirement that causality should be a partial order. Bundle event structures [21] are able to represent OR-causality by allowing each event to be caused by a member of a bundle of events. Asymmetric event structures [5] introduce the notion of weak causality that can model asymmetric conflicts. Inhibitor event structures [4] are able to faithfully capture the dependencies among events which arise in the presence of read and inhibitor arcs. In [6] event structures, where the causality relation may be circular, are investigated, and in [1] the notion of dynamic causality is considered. Configuration structures [13, 15] represent the behaviours of event-oriented models as the collections of their configurations. The culminating point in these studies is a highly expressive concurrency model called event structures for resolvable conflict (RC-structures) [15] corresponding to arbitrary nets without self-loops, under the collective token interpretation.

The association of transition systems with event-oriented models has proved to contribute to studying and solving various problems in the analysis and verification of concurrent systems. It is distinguished two methods of providing transition system semantics with event structures: a *configuration-based* and a *residual-based*. In the first (more 'behavioral') method (see [1, 2, 12–15, 18, 19, 26] among others), states of transition systems are understood as sets of configurations, and state transitions are built by starting with the empty configuration and enlarging configurations by already executed events. In the second (more 'structural') method [3, 9, 10, 19–22, 25], states are understood as event structures, and transitions are built by starting with the given event structure as an initial state and removing already executed (or conflicting) parts thereof in the course of an execution. In the literature, the two semantics have occasionally been used alongside each other (see [19] as an example), but their general relationship has not been studied too deeply. In a seminal paper, viz. [23], *bisimulations* between configuration-based and residual-based transition systems have been proved to exist for prime event structures [27]. This result has been extended in [7] to more complex event structure models — prime/bundle/dual event structures with asymmetric conflict. A crucial technical subtlety pertains to the *removal operator* that lies at the heart of residual semantics. Counterexamples illustrate that an isomorphism cannot be achieved with the various removal operators defined in [7, 23]. The paper [8] demonstrates that, nevertheless, the removal operators can be tightened in such a way that *isomorphisms*, rather than just bisimulations, between the two types of transition systems constructed from a single event structure can be obtained within a wide range of event-oriented models (namely, extended prime event structures, bundle/dual event structures, flow event structures, stable/general event structures, configuration structures), in different true concurrent semantics. In the paper [17], the relationships between transition system semantics for RC-structures presented in standard form have been established in step semantics, because only this semantics has been developed in [15].

The aim of this paper is threefold: to define partial order semantics within the configurations of RC-structures, to find a maximal subclass of the models where partial orders work properly, and also to understand how transition systems based on configurations and residuals of RC-structures, presented not necessary in standard form, are related in the context of partial order multiset and step semantics.

This paper is structured as follows. In the next section, we first recall the definitions of the structure and behavior (configurations) of RC-structures, and then consider and study their special properties. In addition, there are defined partial order multiset and step semantics for RC-structures. In Section 3, we determine and investigate a removal operator for RC-structures in order to obtain their residuals. Section 4 contains the definitions of configuration- and residual-based transition systems and provides isomorphism results between them in the semantics under study, within RC-structures, presented not necessary in standard form. Section 5 concludes.

2. Event Structures for Resolvable Conflict

2.1. Basic Definitions of RC-structures and their Properties

In this subsection, we deal with event structures for resolvable conflict that were put forward in [14] to give semantics to general Petri nets. A resolvable conflict structure consists of a set of events and an enabling relation $\vdash$ between sets of events. The enabling $X \vdash Y$ with sets X and Y imposes restrictions on the occurrences of events in Y by requiring that for all events in Y to occur, their *causes* – the events in X – have to occur before. This allows for modeling the case when a and b cannot occur together until c occurs, i.e., initially a and b are in conflict until the occurrence of c *resolves this conflict.* In resolvable conflict structures, the enabling relation can also model conflicts: events from a set Y are *in irresolvable conflict* iff there is no enabling of the form $X \vdash Y$ for any set X of events. Further, an event can be *impossible* (i.e. non-executable in any system's run) if it has no enabling or has infinite causes or has impossible causes/predecessors. In [15, 17], strict interrelations of resolvable conflict structures have been established with a variety of event-oriented models known from the literature that are unable to model the phenomena of resolvable conflict.

Definition 1. *An* event structure for resolvable conflict (RC-structure) over L *is a tuple* $\mathcal{E} = (E, \vdash, L, l)$*, where* E *is a set of events;* $\vdash \subseteq \mathcal{P}(E) \times \mathcal{P}(E)$ *is the* enabling relation*;* L *is a set of labels;* $l : E \to L$ *is a labeling function.*

Introduce auxiliary notions and notations. For an RC-structure $\mathcal{E}$ over L, a subset $X \subseteq E$, and events $e, d \in E$, define:

- $Con(X) \iff \forall \widehat{X} \subseteq X \exists Z \subseteq E\colon Z \vdash \widehat{X}$ (*consistency*);
- $e \sharp d \iff \neg Con(\{d, e\})$ (*conflict*);
- $e \prec d \iff e \in X$ for all $X \subseteq E$ such that $X \vdash \{d\}$ (*direct causality*);
- X is *left-closed* iff X is finite, and for all $\widetilde{X} \subseteq X$ there exists a set $\widehat{X} \subseteq X$ such that $\widehat{X} \vdash \widetilde{X}$. The set of the left-closed sets of $\mathcal{E}$ is denoted as $LC(\mathcal{E})$. Clearly, we have $Con(X)$ for all $X \in LC(\mathcal{E})$;
- X is a *configuration of* $\mathcal{E}$ iff X can be represented as an ordered set $\{e_1, \ldots, e_n\}$ $(n \geq 0)$ such that for all $i \leq n$ and for all $Y \subseteq \{e_1, \ldots, e_i\}$, there is $Z \subseteq \{e_1, \ldots, e_{i-1}\}$ such that $Z \vdash Y$. Let $Conf(\mathcal{E})$ be the set of configurations of $\mathcal{E}$. Clearly, any configuration X is a left-closed set but not conversely. An event $e \in E$ is called *impossible* (non-executable) in $\mathcal{E}$ if it does not occur in any $X \in Conf(\mathcal{E})$;
- for $X' \subseteq X \in Conf(\mathcal{E})$ and $e, d \in X$, $e \prec_{X'} d \iff e \in Y$ for all $Y \subseteq X'$ such that $Y \vdash \{d\}$ (*direct causality within* X'). Let $\preceq_{X'}$ be the reflexive and transitive closure of $\prec_{X'}$. In case when specifying $\mathcal{E}$ is important for the context, we write $\preceq_X^{\mathcal{E}}$. To save space in the graphical representation, causality between events a and b within a configuration is indicated by the notation $a; b$, the absence of causality (independence) — through $a \| b$.
- for $X, X' \in Conf(\mathcal{E})$, $X \to X'$ iff $X = \{e_1, \ldots, e_m\}$ $(m \geq 0)$, $X' = \{e_1, \ldots, e_m, \ldots, e_n\}$ $(n \geq 0)$, and $m \leq n$.

Lemma 1. *Given an RC-structure $\mathcal{E} = (E, \vdash, L, l)$ and a configuration $X \in Conf(\mathcal{E})$, $\preceq_X$ is a partial order.*

Proof. Suppose that $\mathcal{E} = (E, \vdash, L, l)$ is an RC-structure and $X \in Conf(\mathcal{E})$. As $\preceq_X$ is the transitive and reflexive closure of $\prec_X$, it is sufficient to show that $\preceq_X$ is antisymmetric. Assume $a \preceq_X b$ and $b \preceq_X a$. This means that in X there exist events $e_1, \ldots, e_k$ $(k \geq 1)$ and events $d_1, \ldots, d_l$ $(l \geq 1)$ such that $a = e_1 \prec_X e_2 \ldots e_{k-1} \prec_X e_k = b$ and $b = d_1 \prec_X d_2 \ldots d_{l-1} \prec_X d_l = a$. Since $X \in Conf(\mathcal{E})$, we have an ordered set $X = \{x_1, \ldots, x_m\}$ such that for all $i \leq m$ and all $Y \subseteq \{x_1, \ldots, x_i\}$ there is $Z \subseteq \{x_1, \ldots, x_{i-1}\}$ such that $Z \vdash Y$. W.l.o.g. assume $a = x_p$ and $b = x_q$ $(1 \leq p, q \leq m)$. If $p = q$, the result is obtained. Check the case when $p \neq q$. Consider the sequence $a = e_1 \prec_X e_2 \ldots e_{k-1} \prec_X e_k = b$. As $b = x_q$, we can find $C^b \subseteq \{x_1, \ldots, x_{q-1}\}$ such that $C^b \vdash \{b\}$. By the definition of $\prec_X$, we obtain $e_{k-1} \in C^b \subseteq \{x_1, \ldots, x_{q-1}\}$. Repeating the above reasoning, we come to the conclusion $e_1 = a \in \{x_1, \ldots, x_{q-1}\}$. Hence, $p < q$. Next, consider the sequence $b = d_1 \prec_X d_2 \ldots d_{l-1} \prec_X d_l = a$. Applying the reasoning analogous to the above, we obtain the contradiction $q < p$.

Thus, $p = q$, i.e. $a = b$. □

The next definition gives structural properties of RC-structures which, in suitable combinations, determine subclasses of the models.

Definition 2. *An RC-structure $\mathcal{E} = (E, \vdash, L, l)$ is:*

- rooted *iff* $(\emptyset, \emptyset) \in \vdash$;
- pure *iff* $X \vdash Y \Rightarrow X \cap Y = \emptyset$;
- singular *iff* $X \vdash Y \Rightarrow X = \emptyset \vee \mid Y \mid = 1$;
- locally conjunctive *iff* $X_i \vdash Y (i \in I \neq \emptyset) \wedge Con(\bigcup_{i \in I} X_i \cup Y) \Rightarrow \bigcap_{i \in I} X_i \vdash Y$;
- in standard form *iff* $\vdash = \{(A, B) \mid A \cap B = \emptyset,\ A \cup B \in LC(\mathcal{E})\}$.

Some of the definitions above lead to the following

Observation. Given an RC-structure $\mathcal{E} = (E, \vdash, L, l)$,

(i) $\emptyset \in LC(\mathcal{E})$ iff $\mathcal{E}$ is rooted;

(ii) $Conf(\mathcal{E}) = \emptyset$ iff $\mathcal{E}$ is not rooted;

(iii) $\mathcal{E}$ is pure, if $\mathcal{E}$ is in standard form.

Example 1. *First, consider the RC-structure $\mathcal{E}_1 = (E_1, \vdash_1, L, l_1)$ from [15], where $E_1 = \{a, b, c\}$; $\vdash_1$ consists of $\emptyset \vdash_1 X$ for all $X \neq \{a, b\}$ and $\{c\} \vdash_1 \{a, b\}$; $L = E_1$; and l_1 is the identity labeling function. It is easy to see that $LC(\mathcal{E}_1) = Conf(\mathcal{E}_1) = \{\emptyset,\ \{a\},\ \{b\},\ \{c\},\ \{a, c\},\ \{b, c\},\ \{a, b, c\}\}$. In addition, we get $\prec_X^{\mathcal{E}_1} = \emptyset$, for all $X \in Conf(\mathcal{E}_1)$, and, for example, $\{b\} \rightarrow \{b, c, a\}$. This RC-structure models the initial conflict between the events a and b that can be resolved by the occurrence of the event c. The RC-structure is:*

- *rooted (because $(\emptyset, \emptyset) \in \vdash_1$),*
- *pure (because $X \cap Y = \emptyset$, for all $(X, Y) \in \vdash_1$),*
- *not singular (because $(\{c\}, \{a, b\}) \in \vdash_1$),*
- *locally conjunctive (because X is unique, for all $(X, Y) \in \vdash_1$)).*
- *not in standard form (see the standard form of $\mathcal{E}_1$ in Example 3).*

Second, consider the RC-structure $\mathcal{E}_2 = (E_2, \vdash_2, L, l_2)$, where $E_2 = \{a, b, c\}$; $\vdash_2 = \{(\emptyset, \emptyset),\ (\emptyset, \{a\}),\ (\{a\}, \{b\}),\ (\{a\}, \{c\}),\ (\{b\}, \{c\}),\ (\emptyset, \{a, b\}),\ (\emptyset, \{a, c\}),\ (\emptyset,\ \{b, c\}),\ (\emptyset,\ \{a, b, c\})\}$; $L = E_2$; and l_2 is the identity labeling function. Using the relation $\vdash_2$, we get that $LC(\mathcal{E}_2) = Conf(\mathcal{E}_2) = \{\emptyset,\ \{a\},\ \{a, b\},\ \{a, c\},\ \{a, b, c\}\}$. In addition, $\prec_{\{a,b\}}^{\mathcal{E}_2} = \{(a, b)\}$, $\prec_{\{a,c\}}^{\mathcal{E}_2} = \{(a, c)\}$,

$\prec^{\mathcal{E}_2}_{\{b,c\}} = \{(b,c)\}$, $\prec^{\mathcal{E}_2}_{\{a,b,c\}} = \{(a,b)\}$, *and, for example,* $\{a\} \to \{a,b,c\}$. *The* RC*-structure is:*

- *rooted (because* $(\emptyset,\emptyset) \in \vdash_2$*),*
- *pure (because* $X \cap Y = \emptyset$*, for all* $(X,Y) \in \vdash_2$*),*
- *singular (because* $X = \emptyset \vee \mid Y \mid = 1$*, for all* $(X,Y) \in \vdash_2$*),*
- *not locally conjunctive (because* $(\{a\},\{c\})$*,* $(\{b\},\{c\}) \in \vdash_2$ *and* $(\emptyset,\{c\}) \notin \vdash_2$*),*
- *not in standard form (see the standard form of* $\mathcal{E}_2$ *in Example 2).*

Third, examine the RC*-structure* $\mathcal{E}_3 = (E_3, \vdash_3, L, l_3)$*, where* $E_3 = \{a,b\}$*;* $\vdash_3 = \{(\emptyset,\emptyset)$*,* $(\emptyset,\{a\})$*,* $(\emptyset,\{b\})$*,* $(\{a\},\{a,b\})\}$*;* $L = E_3$*; and* l_3 *is the identity labeling function. It is easy to see that* $LC(\mathcal{E}_3) = Conf(\mathcal{E}_3) = \{\emptyset,\ \{a\},\ \{b\},\ \{a,b\}\}$*. It is not difficult to observe that* $\prec^{\mathcal{E}_3}_{\{a,b\}} = \emptyset$*. Notice that* $\{a\} \to \{a,b\}$ *and* $\{b\} \not\to \{a,b\}$*. The* RC*-structure is:*

- *rooted (because* $(\emptyset,\emptyset) \in \vdash_3$*),*
- *not pure (because* $(\{a\},\{a,b\}) \in \vdash_3$*),*
- *not singular (because* $(\{a\},\{a,b\}) \in \vdash_3$*),*
- *locally conjunctive (because* X *is unique, for all* $(X,Y) \in \vdash_3$*)),*
- *not in standard form (see the standard form of* $\mathcal{E}_3$ *in Example 2).*

Fourth, check the RC*-structure* $\mathcal{E}_4 = (E_4, \vdash_4, L, l_4)$*, where* $E_4 = \{a\}$*;* $\vdash_4 = \{(\emptyset,\{a\})$*,* $(\{a\},\emptyset)\}$*;* $L = E_4$*; and* l_4 *is the identity labeling function. It is easy to see that* $LC(\mathcal{E}_4) = \{\{a\}\}$ *and* $Conf(\mathcal{E}_4) = \emptyset$*. The* RC*-structure is:*

- *not rooted (because* $(\emptyset,\emptyset) \notin \vdash_4$*),*
- *pure (because* $X \cap Y = \emptyset$*, for all* $(X,Y) \in \vdash_4$*),*
- *not singular (because* $(\{a\},\emptyset) \in \vdash_4$*),*
- *locally conjunctive (because* X *is unique, for all* $(X,Y) \in \vdash_4$*)),*
- *in standard form.*

Fifth, observe $\mathcal{E}_5 = (E_5, \vdash_5, L, l_5)$*, where* $E_5 = \{a,b,c\}$*;* $\vdash_5 = \{(\emptyset,\emptyset)$*,* $(\emptyset,\{a\})$*,* $(\emptyset,\{b\})$*,* $(\{a\},\{c\})$*,* $(\{b\},\{c\})$*,* $(\emptyset,\{a,c\})$*,* $(\emptyset,\{b,c\})\}$*;* $L = E_5$*; and* l_5 *is the identity labeling function. It is not difficult to see that* $LC(\mathcal{E}_5) = Conf(\mathcal{E}_5) = \{\emptyset,\ \{a\},\ \{b\},\ \{a,c\},\ \{b,c\}\}$*. Also,* $\prec^{\mathcal{E}_5}_{\{a,c\}} = \{(a,c)\}$*,* $\prec^{\mathcal{E}_5}_{\{b,c\}} = \{(b,c)\}$*, and* $\{a\} \to \{a,c\}$ *and* $\{b\} \to \{b,c\}$*. The* RC*-structure is:*

- *rooted (because* $(\emptyset,\emptyset) \in \vdash_5$*),*
- *pure (because* $X \cap Y = \emptyset$*, for all* $(X,Y) \in \vdash_5$*),*
- *singular (because* $X = \emptyset \vee \mid Y \mid = 1$*, for all* $(X,Y) \in \vdash_5$*),*
- *locally conjunctive (because* $(\{a\},\{c\})$*,* $(\{b\},\{c\})$*, and* $\neg(Con(\{a,b,c\}))$*),*
- *not in standard form (see the standard form of* $\mathcal{E}_5$ *in Example 2).*

Next, we establish essential facts concerning the standard form of RC-structures.

Proposition 1. *Given an RC-structure* $\mathcal{E} = (E, \vdash, L, l)$,

(i) $\mathcal{E}$ *can be transformed into the RC-structure* $SF(\mathcal{E}) = (E, \widetilde{\vdash}, L, l)$ *in standard form such that* $LC(\mathcal{E}) = LC(SF(\mathcal{E}))$. *Moreover,* $SF(\mathcal{E})$ *is rooted iff* $\mathcal{E}$ *is rooted;*

(ii) $\mathit{Conf}(\mathcal{E}) = \mathit{Conf}(SF(\mathcal{E}))$, *and, moreover,* $X \to X'$ *in* $\mathcal{E} \iff X \to X'$ *in* $SF(\mathcal{E})$, *if* $\mathcal{E}$ *is a pure RC-structure;*

(iii) for a configuration $X \in \mathit{Conf}(\mathcal{E})$, $\preceq_X^{\mathcal{E}} = \preceq_X^{SF(\mathcal{E})}$, *if* $\mathcal{E}$ *is a pure, singular and locally conjunctive RC-structure.*

Proof. (i) For the transformation $\mathcal{E} = (E, \vdash, L, l)$ into standard form $SF(\mathcal{E}) = (E, \widetilde{\vdash}, L, l)$, we can directly put $\widetilde{\vdash} = \{(A, B) \mid B \subseteq C \in LC(\mathcal{E}), A = C \setminus B\}$.

Clearly, $LC(\mathcal{E}) = \emptyset$ iff $LC(SF(\mathcal{E})) = \emptyset$. Suppose $X \in LC(\mathcal{E})$. For any $Y \subseteq X$, take $Z = X \setminus Y$. Then, $Z \,\widetilde{\vdash}\, Y$, by the construction of $\widetilde{\vdash}$. Hence, $Z \cup Y = X \in LC(SF(\mathcal{E}))$. Conversely, assume $X \in LC(SF(\mathcal{E}))$. Then, there is $Z \subseteq X$ such that $Z \,\widetilde{\vdash}\, X$. By the construction of $\widetilde{\vdash}$, $X = Z \cup X \in LC(\mathcal{E})$. Thus, $LC(\mathcal{E}) = LC(SF(\mathcal{E}))$. It is easy to see that $SF(\mathcal{E})$ is rooted iff $\mathcal{E}$ is rooted.

(ii) Assume $\mathcal{E}$ being a pure RC-structure. Using Observation (ii) and item (i), it is straightforward to show that $\mathit{Conf}(\mathcal{E}) = \emptyset$ iff $\mathit{Conf}(SF(\mathcal{E})) = \emptyset$. Take an arbitrary $X \in \mathit{Conf}(SF(\mathcal{E}))$. This means that $X = \{e_1, \ldots, e_n\}$ $(n \geq 0)$ such that for all $i \leq n$ and for all $Y \subseteq \{e_1, \ldots, e_i\}$, there is $Z \subseteq \{e_1, \ldots, e_{i-1}\}$ such that $Z \,\widetilde{\vdash}\, Y$. Clearly, $X_i = \{e_1, \ldots, e_i\} \in LC(SF(\mathcal{E}))$, for all $i \leq n$. Thanks to item (i), $X_i \in LC(\mathcal{E})$, for all $i \leq n$. Take arbitrary $i \leq n$ and $Y \subseteq X_i$. Consider two possible cases.

$e_i \in Y$. Since $X_i \in LC(\mathcal{E})$, there is $Z \subseteq X_i$ such that $Z \vdash Y$. Due to $\mathcal{E}$ being a pure RC-structure, $Z \cap Y = \emptyset$. Hence, $e_i \notin Z$. So, $Z \subseteq X_{i-1}$.

$e_i \notin Y$. This means that $Y \subseteq X_{i-1}$. Since $X_{i-1} \in LC(\mathcal{E})$, there is $Z \subseteq X_{i-1}$ such that $Z \vdash Y$.

Thus, $X \in \mathit{Conf}(\mathcal{E})$.

Suppose $X \in \mathit{Conf}(\mathcal{E})$. By Observation (iii), $SF(\mathcal{E})$ is a pure RC-structure. Following similar lines of the proof in the opposite direction, we obtain $X \in \mathit{Conf}(SF(\mathcal{E}))$.

Check that $X \to X'$ in $\mathcal{E} \iff X \to X'$ in $SF(\mathcal{E})$. Assume that $X \to X'$ in $\mathcal{E}$. This means that $X,\ X' \in \mathit{Conf}(\mathcal{E})$, $X = \{e_1, \ldots, e_k\}$ and $X' = \{e_1, \ldots, e_n\}$ $(0 \leq k \leq n)$. Due to $\mathcal{E}$ being a pure RC-structure, we may conclude that $X,\ X' \in \mathit{Conf}(SF(\mathcal{E}))$ with the same ordering of events as shown above. Hence, $X \to X'$ in $SF(\mathcal{E})$. Conversely, suppose that $X \to X'$ in $SF(\mathcal{E})$. The proof of the result is analogous to that in the opposite direction. Thus, $X \to X'$

in $\mathcal{E}$.

(iii) According to items (i) and (ii), we have that $LC(\mathcal{E}) = LC(SF(\mathcal{E}))$ and $Conf(\mathcal{E}) = Conf(SF(\mathcal{E}))$, respectively, since $\mathcal{E}$ is a pure RC-structure.

Consider arbitrary events $e, d \in X \in Conf(\mathcal{E})$ such that $e \prec_X^{\mathcal{E}} d$. Check that $e \prec_X^{SF(\mathcal{E})} d$. Take an arbitrary set $Z \subseteq X$ such that $Z \vdash_{SF(\mathcal{E})} \{d\}$. By the construction of $SF(\mathcal{E})$, we obtain $Z \cup \{d\} \in LC(\mathcal{E})$. This means that for all $Z' \subseteq Z \cup \{d\}$ there exists a set $Z'' \subseteq Z \cup \{d\}$ such that $Z'' \vdash_{\mathcal{E}} Z'$. Hence, we can find a set $W \subseteq Z \cup \{d\}$ such that $W \vdash_{\mathcal{E}} \{d\}$. Note that $W \cap \{d\} = \emptyset$, due to $\mathcal{E}$ being a pure RC-structure. So, $W \subseteq Z$. By the definition of $\prec_X^{\mathcal{E}}$, we have that $e \in W$. Hence, $e \prec_X^{SF(\mathcal{E})} d$.

Conversely, take arbitrary events $e, d \in X$ such that $e \prec_X^{SF(\mathcal{E})} d$. We need to show that $e \preceq_X^{\mathcal{E}} d$. Since $X \in Conf(\mathcal{E})$, we have an ordered set $X = \{e_1, \ldots, e_m\}$ such that for all $i \leq m$ and all $Y \subseteq \{e_1, \ldots, e_i\}$ there is $Z \subseteq \{e_1, \ldots, e_{i-1}\}$ such that $Z \vdash_{\mathcal{E}} Y$. W.l.o.g. assume $d = e_q$ $(1 \leq q \leq m)$. Clearly, $X_q = \{e_1, \ldots, e_q\} \in LC(\mathcal{E})$ and $Con(X_q)$. Define the set $C^d = \{x \in X_q \mid x \preceq_X^{\mathcal{E}} d\}$. Check that $C^d \in LC(\mathcal{E})$. Take an arbitrary $W \subseteq C^d$. Treat three admissible cases.

$|W| = 0$. Since $X \in Conf(\mathcal{E})$, it holds that $Conf(\mathcal{E}) \neq \emptyset$. By Observation (ii), we get that $\mathcal{E}$ is a rooted RC-structure. Hence, we have $\emptyset \vdash_{\mathcal{E}} \emptyset = W$.

$|W| = 1$. W.l.o.g. assume $W = \{z\}$. As $W \subseteq X_q \in LC(\mathcal{E})$, there is at least one $A_i \subseteq X_q$ such that $A_i \vdash_{\mathcal{E}} \{z\}$. Due to $\mathcal{E}$ being a locally conjunctive RC-structure and $Con(X_q)$, we can find $A = \bigcap_{i \in I} A_i \subseteq X_q$ such that $A \vdash_{\mathcal{E}} \{z\}$. Then, we obtain that $a \prec_X^{\mathcal{E}} z$ for all $a \in A$, thanks to the definition of $\prec_X^{\mathcal{E}}$. Because of $z \in C^d$, we have $z \preceq_X^{\mathcal{E}} d$. Hence, for all $a \in A$ it holds $a \preceq_X^{\mathcal{E}} d$, by the transitivity of $\preceq_X^{\mathcal{E}}$. Thus, $A \subseteq C^d$.

$|W| \geq 2$. As $W \subseteq X_q \in LC(\mathcal{E})$, we can find $W' \subseteq X_q$ such that $W' \vdash_{\mathcal{E}} W$. Since $\mathcal{E}$ is a singular RC-structure and $|W| \geq 2$, it holds that $W' = \emptyset \subseteq C^d$.

Thus, $C^d \in LC(\mathcal{E})$ and $d \in C^d$. Due to the construction of $SF(\mathcal{E})$, we obtain $C^d \setminus \{d\} \vdash_{SF(\mathcal{E})} \{d\}$. As $e \prec_X^{SF(\mathcal{E})} d$, it holds $e \in C^d \setminus \{d\}$. This means that $e \preceq_X^{\mathcal{E}} d$, by the definition of C^d. $\square$

We illustrate the validity of Proposition 1.

Example 2. *First, consider the rooted, pure, not locally conjunctive and singular RC-structure* $\mathcal{E}_2 = (E_2, \vdash_2, L, l_2)$ *from Example 1. Recall that* $E_2 = \{a, b, c\}$; $\vdash_2 = \{(\emptyset, \emptyset), (\emptyset, \{a\}), (\{a\}, \{b\}), (\{a\}, \{c\}), (\{b\}, \{c\}), (\emptyset, \{a, b\}), (\emptyset, \{a, c\}), (\emptyset, \{b, c\}), (\emptyset, \{a, b, c\})\}$; $L = E_2$; *and* l_2 *is the identity labeling function. We know from Example 1 that* $LC(\mathcal{E}_2) = Conf(\mathcal{E}_2) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$, *and* $\prec_{\{a,b,c\}}^{\mathcal{E}_2} = \{(a, b)\}$.

The structure $\mathcal{E}_2$ can be presented in standard form $\widetilde{\mathcal{E}}_2$, with $\widetilde{\vdash}_2 = \{(\emptyset,\emptyset),\ (\emptyset,\{a\}),\ (\{a\},\emptyset),\ (\{a\},\{b\}),\ (\{b\},\{a\}),\ (\emptyset,\{a,b\}),\ (\{a,b\},\emptyset),\ (\{a\},\{c\}),\ (\{c\},\ \{a\}),\ (\emptyset,\ \{a,c\}),\ (\{a,c\},\ \emptyset),\ (\{a\},\{b,c\}),\ (\{b,c\},\{a\}),\ (\{b\},\ \{a,c\}),\ (\{a,c\},\ \{b\}),\ (\{c\},\ \{a,b\}),\ (\{a,b\},\ \{c\}),\ (\emptyset,\ \{a,b,c\}),\ (\{a,b,c\},\ \emptyset)\}$. It is easy to see that $LC(\mathcal{E}_2) = LC(\widetilde{\mathcal{E}}_2)$, and $Conf(\mathcal{E}_2) = Conf(\widetilde{\mathcal{E}}_2)$, as $\mathcal{E}_2$ is pure. Using the relation $\widetilde{\vdash}_2$, we get $\prec^{\widetilde{\mathcal{E}}_2}_{\{a,b,c\}} = \{(a,b),(a,c)\}$. Then, $\prec^{\mathcal{E}_2}_{\{a,b,c\}} \neq \prec^{\widetilde{\mathcal{E}}_2}_{\{a,b,c\}}$, because $\mathcal{E}_2$ is not locally conjunctive.

Second, examine the rooted, not pure, locally conjunctive and not singular RC-structure $\mathcal{E}_3 = (E_3, \vdash_3, L, l_3)$ from Example 1. Here, $E_3 = \{a,b\}$; $\vdash_3 = \{(\emptyset,\emptyset),\ (\emptyset,\{a\}),\ (\emptyset,\{b\}),\ (\{a\},\{a,b\})\}$; $L = E_3$; and l_3 is the identity labeling function. We know from Example 1 that $LC(\mathcal{E}_3) = Conf(\mathcal{E}_3) = \{\emptyset,\ \{a\},\ \{b\},\ \{a,b\}\}$, and $\prec^{\mathcal{E}_3}_{\{a,b\}} = \emptyset$. In addition, $\{a\} \to \{a,b\}$ and $\{b\} \not\to \{a,b\}$ in $\mathcal{E}_3$.

The structure $\mathcal{E}_3$ can be presented in standard form $\widetilde{\mathcal{E}}_3$, with $\widetilde{\vdash}_3 = \{(\emptyset,\emptyset),\ (\emptyset,\{a\}),\ (\{a\},\emptyset),\ (\emptyset,\{b\}),\ (\{b\},\emptyset),\ (\{a\},\{b\}),\ (\{b\},\{a\}),\ (\emptyset,\{a,b\}),\ (\{a,b\},\emptyset)\}$. It is not difficult to see that $Conf(\mathcal{E}_3) = \{\emptyset,\ \{a\},\ \{b\},\ \{a,b\}\}$, and $\prec^{\widetilde{\mathcal{E}}_3}_{\{a,b\}} = \emptyset$. Also it is true that $\{a\} \to \{a,b\}$ and, however, $\{b\} \to \{a,b\}$ in $\widetilde{\mathcal{E}}_3$, because $\mathcal{E}_3$ is not pure RC-structure.

Third, examine the rooted, pure, locally conjunctive and singular RC-structure $\mathcal{E}_5 = (E_5, \vdash_5, L, l_5)$ from Example 1. Recall that $E_5 = \{a,b,c\}$; $\vdash_5 = \{(\emptyset,\emptyset),\ (\emptyset,\{a\}),\ (\emptyset,\{b\}),\ (\{a\},\{c\}),\ (\{b\},\{c\}),\ (\emptyset,\{a,c\}),\ (\emptyset,\{b,c\})\}$; $L = E_5$; and l_5 is the identity labeling function. We know that $LC(\mathcal{E}_5) = Conf(\mathcal{E}_5) = \{\emptyset,\ \{a\},\ \{b\},\ \{a,c\},\ \{b,c\}\}$, and $\prec^{\mathcal{E}_5}_{\{a,c\}} = \{(a,c)\}$, $\prec^{\mathcal{E}_5}_{\{b,c\}} = \{(b,c)\}$.

The structure $\mathcal{E}_5$ can be presented in standard form $\widetilde{\mathcal{E}}_5$, with $\widetilde{\vdash}_5 = \{(\emptyset,\emptyset),\ (\emptyset,\{a\}),\ (\{a\},\emptyset),\ (\emptyset,\{b\}),\ (\{b\},\emptyset),\ (\{a\},\{c\}),\ (\{c\},\ \{a\}),\ (\emptyset,\ \{a,c\}),\ (\{a,c\},\ \emptyset),\ (\{b\},\{c\}),\ (\{c\},\ \{b\}),\ (\emptyset,\ \{b,c\}),\ (\{b,c\},\ \emptyset)\}$. It is easy to see that $LC(\mathcal{E}_5) = LC(\widetilde{\mathcal{E}}_5)$, and $Conf(\mathcal{E}_5) = Conf(\widetilde{\mathcal{E}}_5)$, as $\mathcal{E}_5$ is pure. Using the relation $\widetilde{\vdash}_5$, we get $\prec^{\widetilde{\mathcal{E}}_5}_{\{a,c\}} = \{(a,c)\}$ and $\prec^{\widetilde{\mathcal{E}}_5}_{\{b,c\}} = \{(b,c)\}$. Then, $\prec^{\mathcal{E}_5}_{\{a,c\}} = \prec^{\widetilde{\mathcal{E}}_5}_{\{a,c\}}$ and $\prec^{\mathcal{E}_5}_{\{b,c\}} = \prec^{\widetilde{\mathcal{E}}_5}_{\{b,c\}}$, because $\mathcal{E}_5$ is a rooted, pure, locally conjunctive and singular RC-structure.

2.2. Different Semantics for RC-structures

In this subsection, we introduce partial order multiset (pom) and step (step) semantics for RC-structures.

We first define auxiliary notions and notations. For configurations $X, X' \in Conf(\mathcal{E})$, we write:

- $X \to_{pom} X'$ iff $X \to X'$;
- $X \to_{step} X'$ iff $X \to X'$ in $\mathcal{E}$, and $X'' \in Conf(\mathcal{E})$, for all $X \subseteq X'' \subseteq X'$.

For $\star \in \{pom, step\}$, a configuration $X \in Conf(\mathcal{E})$ is a *configuration in $\star$-semantics* of $\mathcal{E}$ iff $\emptyset \rightarrow_\star^* X$, where $\rightarrow_\star^*$ is the reflexive and transitive closure of $\rightarrow_\star$. Let $Conf_\star(\mathcal{E})$ denote the set of configurations in $\star$-semantics of $\mathcal{E}$.

Lemma 2. *Given an RC-structure $\mathcal{E}$ and $\star \in \{pom, step\}$,*

(i) $Conf_\star(\mathcal{E}) = Conf(\mathcal{E})$;

(ii) $X \rightarrow_\star X'$ in $\mathcal{E}$ $\iff$ $X \rightarrow_\star X'$ in $SF(\mathcal{E})$, if $\mathcal{E}$ is a pure RC-structure.

Proof. (i) Due to the definition of $\star$-semantics, we have $Conf_\star(\mathcal{E}) \subseteq Conf(\mathcal{E})$. We shall show $Conf(\mathcal{E}) \subseteq Conf_\star(\mathcal{E})$. Take an arbitrary $X \in Conf(\mathcal{E})$. This means that X can be represented as an ordered set $\{e_1, \ldots, e_n\}$ $(n \geq 0)$ such that for all $i \leq n$ and for all $Y \subseteq \{e_1, \ldots, e_i\}$, there is $Z \subseteq \{e_1, \ldots, e_{i-1}\}$ such that $Z \vdash Y$. Clearly, $X_i = \{e_1, \ldots, e_i\} \in Conf(\mathcal{E})$ for all $i \leq n$. If $\mathcal{E}$ is a rooted RC-structure, it holds $\emptyset \in Conf(\mathcal{E})$. Then, we obtain $\emptyset \rightarrow_\star^* X$, by the definitions of a configuration and the relation $\rightarrow_\star$. Hence, $X \in Conf_\star(\mathcal{E})$. If $\mathcal{E}$ is not rooted, it is true that $Conf_\star(\mathcal{E}) = Conf(\mathcal{E}) = \emptyset$.

(ii) It follows from the definitions of the transition relations and Proposition 1(ii). □

For an RC-structure $\mathcal{E}$ and $X, X' \in Conf(\mathcal{E})$, we write $\mathcal{E}\lceil(X' \setminus X) = ((X' \setminus X), \preceq_{(X'\setminus X)} = \preceq_{X'} \cap (X' \setminus X \times X' \setminus X), L, l\mid_{(X'\setminus X)})$, if $X \rightarrow_{pom} X'$. We say that $\mathcal{E}\lceil(X' \setminus X)$ and $\mathcal{E}\lceil(Y' \setminus Y)$ are *isomorphic* iff there is a bijection $\iota : (X' \setminus X) \rightarrow (Y' \setminus Y)$ such that $(x, x') \in \preceq_{(X'\setminus X)}$ iff $(\iota(x), \iota(x')) \in \preceq_{(Y'\setminus Y)}$, for all $x, x' \in (X' \setminus X)$, and $l(x) = l(\iota(x))$, for all $x \in (X' \setminus X)$. Let $[\mathcal{E}\lceil(X' \setminus X)]$ denote the isomorphic class of $\mathcal{E}\lceil(X' \setminus X)$.

Given $\star \in \{pom, step\}$, an RC-structure $\mathcal{E}$ over L, and configurations $X, X' \in Conf_\star(\mathcal{E})$ such that $X \rightarrow_\star X'$, we write:

$$l_\star(X' \setminus X) = \begin{cases} [\mathcal{E}\lceil(X' \setminus X)], & \text{if } \star = pom, \\ M \ (\forall a \in L\colon M(a) = |\{e \in X' \setminus X \mid l(e) = a\}|), & \text{if } \star = step. \end{cases}$$

3. Removal Operator for RC-structures

The standard form of RC-structures and their ability to specify impossible events allow us to develop a relatively simple structural definition of the removal operator which is useful for constructing residuals of RC-structures.

Definition 3. *For an RC-structure $\mathcal{E} = (E, \vdash, L, l)$ in standard form and $X \in LC(\mathcal{E})$, the* *removal operator is defined as follows:* $\mathcal{E} \setminus X = (E', \vdash', L, l')$, *where*

$$\begin{array}{rcl} E' & = & E \setminus X, \\ \vdash' & = & \{(A', B') \mid \exists (A, B) \in \vdash \text{ s.t. } A' = A \cap E', B' = B \cap E', (A' \cup B' \cup X) \in LC(\mathcal{E})\}, \\ l' & = & l \mid_{E'} . \end{array}$$

According to the definition above, we remove all events in X and retain the events that, when combined with X, do not form left-closed sets and, therefore, conflict with some events in X; however, we make the conflicting retained events impossible by deleting their enabling relations.

Consider properties of the removal operator.

Lemma 3. *Given an RC-structure $\mathcal{E}$ in standard form and $X \in Conf(\mathcal{E})$,*

(i) $\mathcal{E} \setminus X$ is an RC-structure;

(ii) $(X \cup Y) \in LC(\mathcal{E}) \iff Y \in LC(\mathcal{E} \setminus X)$, for any $Y \subseteq E'$;

(iii) $\mathcal{E} \setminus X$ is a rooted RC-structure $\mathcal{E}$ in standard form.

Proof. (i) According to Definition 1, $\mathcal{E} \setminus X$ is an RC-structure.

(ii) Take an arbitrary $Y \subseteq E'$. Then, $X \cap Y = \emptyset$, due to Definition 3.

($\Rightarrow$) Suppose $(X \cup Y) \in LC(\mathcal{E})$. Then, for all $\widetilde{Y} \subseteq Y$, $(\widetilde{Y} \cup \widehat{Y}) \in LC(\mathcal{E})$, where $\widehat{Y} = X \cup Y \setminus \widetilde{Y}$. As $\mathcal{E}$ is in standard form, $\widehat{Y} \vdash \widetilde{Y}$, for all $\widetilde{Y} \subseteq Y$ and the corresponding $\widehat{Y}$. Obviously, $(\widetilde{Y} \cup (\widehat{Y}' = \widehat{Y} \setminus X) \cup X) \in LC(\mathcal{E})$ and $\widehat{Y}' \subseteq Y$. Due to the definition of $\vdash'$, for all $\widetilde{Y} \subseteq Y$, there exists $\widehat{Y}' \subseteq Y$ such that $\widehat{Y}' \vdash' \widetilde{Y}$. Thus, $Y \in LC(\mathcal{E} \setminus X)$.

($\Leftarrow$) Assume $Y \in LC(\mathcal{E} \setminus X)$. Then, for Y there is $\widehat{Y} \subseteq Y$ such that $\widehat{Y} \vdash' Y$. By the definition of $\vdash'$, this implies that $(X \cup \widehat{Y} \cup Y) = (X \cup Y) \in LC(\mathcal{E})$.

(iii) We first show that $\mathcal{E} \setminus X$ is in standard form.

($\Rightarrow$) Suppose $A' \vdash' B'$. Then, we can find $A \vdash B$ such that $A' = A \cap E'$, $B' = B \cap E'$ and $(A' \cup B' \cup X) \in LC(\mathcal{E})$, due to the definition of $\vdash'$. Since $\mathcal{E}$ is in standard form, it holds that $A \cap B = \emptyset$. This implies that $A' \cap B' = \emptyset$. Thanks to item (ii), we get that $(A' \cup B') \in LC(\mathcal{E} \setminus X)$.

($\Leftarrow$) Assume $C' \in LC(\mathcal{E} \setminus X)$. Take $B' \subseteq C'$ and $A' = C' \setminus B'$. According to item (ii), $(C' \cup X) = (A' \cup B' \cup X) \in LC(\mathcal{E})$. Moreover, since $(A' \cup X) \cap B' = \emptyset$, we get that $A' \cup X \vdash B'$, due to $\mathcal{E}$ being in standard form. Hence, $A' \vdash' B'$, by the definition of $\vdash'$.

As $X \in LC(\mathcal{E})$ and $\mathcal{E}$ is in standard form, $(\emptyset, \emptyset) \in \vdash'$, i.e. $\mathcal{E} \setminus X$ is rooted. □

Next we establish the relationships between configurations and partial orders in the original RC-structure and its residuals.

Proposition 2. *Given an RC-structure $\mathcal{E}$ in standard form,*

(i) for any $X, X' \in Conf(\mathcal{E})$ such that $X \to X'$,

(a) $X' \setminus X \in Conf(\mathcal{E} \setminus X)$;

(b) $\preceq^{\mathcal{E}\setminus X}_{X'\setminus X} = \preceq^{\mathcal{E}}_{X'} \cap (X'\setminus X \times X'\setminus X)$, if $\mathcal{E}$ is the standard form of a singular RC-structure;

(ii) for any $\mathcal{E}' = \mathcal{E} \setminus X$, with $X \in Conf(\mathcal{E})$, and for any $\mathcal{E}'' = \mathcal{E}' \setminus X'$, with $X' \in Conf(\mathcal{E}')$, $X \to X \cup X'$ in $\mathcal{E}$, and $\mathcal{E}'' = \mathcal{E} \setminus (X \cup X')$.

Proof. (i(a)) Take an arbitrary $X, X' \in Conf(\mathcal{E})$ such that $X \to X'$. As $X \to X'$, we have $X = \{e_1, \ldots, e_m\}$ $(m \geq 0)$, $X' = \{e_1, \ldots, e_m, e_{m+1}, \ldots, e_n\}$ $(n \geq 0)$, and $m \leq n$. Let $X' \setminus X = \{e_{m+1}, \ldots, e_n\}$. Check that $X' \setminus X \in Conf(\mathcal{E} \setminus X)$. Take an arbitrary $i \leq n - m$ and an arbitrary $A \subseteq \{e_{m+1}, \ldots, e_{m+i}\}$. Consider possible cases.

$A = \emptyset$. We have $\emptyset \vdash' \emptyset$, because $\mathcal{E} \setminus X$ is rooted, by Lemma 3(iii).

$A \neq \emptyset$. Notice that $\{e_1, \ldots, e_{m+i}\} \in Conf(\mathcal{E}) \subseteq LC(\mathcal{E})$, for all $i \leq n - m$. Define $B = \{e_1, \ldots, e_{m+i}\} \setminus A$, if $e_{m+i} \in A$, and $B = \{e_1, \ldots, e_{m+i-1}\} \setminus A$, otherwise. Clearly, $B \subseteq \{e_1, \ldots, e_{m+i-1}\}$. Since $\mathcal{E}$ is in standard form, we get $B \vdash A$. According to Lemma 3(ii), it holds $B \setminus X \cup A \in LC(\mathcal{E} \setminus X)$. We know that $\mathcal{E} \setminus X$ is in standard form, thanks to Lemma 3(iii). Hence, $B \setminus X \vdash' A$.

Thus, $X' \setminus X \in Conf(\mathcal{E} \setminus X)$.

(i(b)) Take arbitrary events $e, d \in X' \setminus X$. Assume $e \prec^{\mathcal{E}}_{X'} d$. Check that $e \prec^{\mathcal{E}\setminus X}_{X'\setminus X} d$. Take an arbitrary set $Z \subseteq X'\setminus X$ such that $Z \vdash_{\mathcal{E}\setminus X} \{d\}$. By Definition 3, we obtain $Z \cup \{d\} \cup X \in LC(\mathcal{E})$. This means that for all $Z' \subseteq Z \cup \{d\} \cup X$ there exists a set $Z'' \subseteq Z \cup \{d\} \cup X$ such that $Z'' \vdash_{\mathcal{E}} Z'$. Hence, we can find a set $W \subseteq Z \cup \{d\} \cup X$ such that $W \vdash_{\mathcal{E}} \{d\}$. By Observation (iii), it holds that $\mathcal{E}$ is a pure RC-structure. Thus, $W \cap \{d\} = \emptyset$. So, $W \subseteq Z \cup X \subset X'$. By the definition of $\prec^{\mathcal{E}}_{X'}$, we have $e \in W$. Since $e \in X' \setminus X$ and $W \subseteq Z \cup X$, we conclude $e \in Z$. Hence, $e \prec^{\mathcal{E}\setminus X}_{X'\setminus X} d$.

Conversely, suppose $e \prec^{\mathcal{E}\setminus X}_{X'\setminus X} d$. We need to show that $e \prec^{\mathcal{E}}_{X'} d$. Consider an arbitrary set $Z \subseteq X'$ such that $Z \vdash_{\mathcal{E}} \{d\}$. Due to $\mathcal{E}$ being in standard form, we obtain that $Z \cup \{d\} \in LC(\mathcal{E})$. Moreover, $d \notin Z$, by Observation (iii). As $\mathcal{E} = SF(\mathcal{F})$, we obtain $LC(\mathcal{E}) = LC(\mathcal{F})$, due to Proposition 1(i). It is sufficient to check that $X \cup Z \cup \{d\} \in LC(\mathcal{F})$. Take an arbitrary $W \subseteq X \cup Z \cup \{d\}$. Three cases are admissible.

$|W| = 0$. Since $X' \in Conf(\mathcal{E})$, we may conclude that $Conf(\mathcal{E}) \neq \emptyset$. This means that $\mathcal{E}$ is rooted, by Observation (ii). Due to Proposition 1 (i), we get that $\mathcal{F}$ is a rooted RC-structure. Hence, $\emptyset \vdash_{\mathcal{F}} \emptyset = W$.

$|W| = 1$. W.l.o.g. assume $W = \{z\}$. Consider two possible cases.

$z \in X$. Since $X \in LC(\mathcal{E}) = LC(\mathcal{F})$, there is $W' \subseteq X \subseteq X \cup Z \cup \{d\}$ such that $W' \vdash_{\mathcal{F}} \{z\}$.

$z \notin X$. Hence, $z \in (Z \cup \{d\}) \setminus X$. As $Z \cup \{d\} \in LC(\mathcal{E}) = LC(\mathcal{F})$, we can find $W' \subseteq Z \cup \{d\} \subseteq X \cup Z \cup \{d\}$ such that $W' \vdash_{\mathcal{F}} \{z\}$.

$|W| \geq 2$. Since $W \subseteq X \cup Z \cup \{d\} \subseteq X' \in LC(\mathcal{E}) = LC(\mathcal{F})$, there is $W' \subseteq X'$ such that $W' \vdash_{\mathcal{F}} W$. Due to $\mathcal{F}$ being a singular RC-structure and $|W| \geq 2$, it is true that $W' = \emptyset \subseteq X \cup Z \cup \{d\}$.

Thus, $X \cup Z \cup \{d\} \in LC(\mathcal{E}) = LC(\mathcal{F})$. Due to Definition 3, we obtain $Z \setminus X \vdash_{\mathcal{E} \setminus X} \{d\}$. As $e \prec^{\mathcal{E} \setminus X}_{X' \setminus X} d$, it holds $e \in Z \setminus X$. Thus, $e \prec^{\mathcal{E}}_{X'} d$.

(ii) Assume that $X \in \mathit{Conf}(\mathcal{E})$ and $X' \in \mathit{Conf}(\mathcal{E}')$, where $\mathcal{E}' = \mathcal{E} \setminus X$. Then, we can arrange $X = \{e_1, \ldots, e_n\}$ so that for all $i \leq n$ and for all $Y \subseteq \{e_1, \ldots, e_i\}$, there is $Z \subseteq \{e_1, \ldots, e_{i-1}\}$ such that $Z \vdash Y$, and arrange $X' = \{e'_1, \ldots, e'_m\}$ so that for all $j \leq m$ and for all $Y' \subseteq \{e'_1, \ldots, e'_j\}$, there is $Z' \subseteq \{e'_1, \ldots, e'_{i-1}\}$ such that $Z' \vdash' Y'$. Define $e_{n+1} = e'_1$, …, $e_{n+m} = e'_m$. Take an arbitrary $i \leq n+m$ and an arbitrary $Y \subseteq \{e_1, \ldots, e_i\}$. If $i \leq n$, the result follows from the fact that $X = \{e_1, \ldots, e_n\} \in \mathit{Conf}(\mathcal{E})$. Consider the case when $n < i$. Define $Y' = Y \cap E'$. Due to $X' \in \mathit{Conf}(\mathcal{E}')$, there is $Z' \subseteq \{e'_1, \ldots, e'_{i-1}\}$ such that $Z' \vdash' Y'$. By virtue of Lemma 3(iii), $\mathcal{E}' = \mathcal{E} \setminus X$ is in standard form. Thus, $Z' \cup Y' \in LC(\mathcal{E}')$. According to Lemma 3(ii), we may conclude $Z' \cup Y' \cup X \in LC(\mathcal{E})$. Due to $\mathcal{E}$ being in standard form, this implies $Z \vdash Y$, where $Z = (Z' \cup Y' \cup X) \setminus Y$. Clearly, $Z \subseteq Z' \cup X \subseteq \{e_1, \ldots, e_{i-1}\}$. Hence, $X \cup X' \in \mathit{Conf}(\mathcal{E})$. Obviously, $X = \{e_1, \ldots, e_n\} \to X \cup X' = \{e_1, \ldots, e_n, e'_1, \ldots, e'_m\}$.

Check that $(\mathcal{E} \setminus X) \setminus X' = \mathcal{E} \setminus (X \cup X')$. Define $\mathcal{E}'' = \mathcal{E}' \setminus X'$ and $\widetilde{\mathcal{E}} = \mathcal{E} \setminus (X \cup X')$. We need to show $\mathcal{E}'' = \widetilde{\mathcal{E}}$. By Definition 3, $E'' = E' \setminus X' = E \setminus X \setminus X' = E \setminus (X \cup X') = \widetilde{E}$, and $l'' = l\,|_{E''} = l\,|_{\widetilde{E}} = \widetilde{l}$. Verify that $LC(\mathcal{E}'') = LC(\widetilde{\mathcal{E}})$. Thanks to Lemma 3(iii), $\widetilde{\mathcal{E}}$ and $\mathcal{E}''$ are rooted RC-structures in standard form. Then, $LC(\mathcal{E}'') \neq \emptyset$ and $LC(\widetilde{\mathcal{E}}) \neq \emptyset$. W.l.o.g. take an arbitrary $Y \in LC(\mathcal{E}'')$. According to Lemma 3(ii), $Y \in LC(\mathcal{E}'') \iff Y \cup X' \in LC(\mathcal{E}') \iff Y \cup X' \cup X \in LC(\mathcal{E}) \iff Y \in LC(\mathcal{E} \setminus (X \cup X')) = LC(\widetilde{\mathcal{E}})$. Hence, $\vdash'' = \widetilde{\vdash}$. □

We demonstrate the validity of Proposition 2.

Example 3. *First, consider the rooted, pure, locally conjunctive and not singular RC-structure* $\mathcal{E}_1 = (E_1, \vdash_1, L, l_1)$ *from Example 1. Recall that* $E_1 = \{a, b, c\}$; $\vdash_1$ *consists of* $\emptyset \vdash_1 X$ *for all* $X \neq \{a, b\}$ *and* $\{c\} \vdash_1 \{a, b\}$; $L = E_1$; *and* l_1 *is the identity labeling function. We know from Example 1 that* $LC(\mathcal{E}_1) = \mathit{Conf}(\mathcal{E}_1) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{c, a\}, \{b, c\}, \{b, c, a\}\}$.

By building the standard form of $\mathcal{E}_1$, *we get the enabling relation of* $\widetilde{\mathcal{E}}_1$: $\widetilde{\vdash}_1 = \{(\emptyset, \emptyset), (\emptyset, \{a\}), (\{a\}, \emptyset), (\emptyset, \{b\}), (\{b\}, \emptyset), (\emptyset, \{c\}), (\{c\}, \emptyset), (\emptyset, \{a, c\}), (\{a, c\}, \emptyset), (\{a\}, \{c\}), (\{c\}, \{a\}), (\emptyset, \{b, c\}), (\{b, c\}, \emptyset), (\{b\}, \{c\}), (\{c\}, \{b\}), (\{a\}, \{b, c\}), (\{b, c\}, \{a\}), (\{b\}, \{a, c\}),$

$(\{a,c\},\ \{b\})$, $(\{c\},\ \{a,b\})$, $(\{a,b\},\ \{c\})$, $(\emptyset,\ \{a,b,c\})$, $(\{a,b,c\},\ \emptyset)\}$. *It is easy to see that* $LC(\mathcal{E}_1) = LC(\widetilde{\mathcal{E}}_1)$ *and* $Conf(\mathcal{E}_1) = Conf(\widetilde{\mathcal{E}}_1)$. *In addition, we have that* $\{b\} \to \{b,c\}$ *and* $\{b\} \to \{b,c,a\}$ *in* $\widetilde{\mathcal{E}}_1$. *Moreover, it holds* $\prec^{\widetilde{\mathcal{E}}_1}_{\{b,c,a\}} = \emptyset$.

Construct the RC-structure $\widetilde{\mathcal{E}}_1 \setminus \{b\} = (\{a,c\}, \vdash^*_1, L, l^*_1 = l_1|_{\{a,c\}})$, *where* $\vdash^*{}_1 = \{(\emptyset,\emptyset)$, $(\emptyset,\{c\})$, $(\{c\},\emptyset)$, $(\emptyset,\{a,c\})$, $(\{a,c\},\emptyset)$, $(\{a\},\{c\})$, $(\{c\},\ \{a\})\}$. *Then, we obtain* $Conf(\widetilde{\mathcal{E}}_1 \setminus \{b\}) = \{\emptyset,\ \{c\},\ \{c,a\}\}$. *So,* $\{b,c,a\} \setminus \{b\} = \{c,a\} \in Conf(\widetilde{\mathcal{E}}_1 \setminus \{b\})$. *On the other hand,* $\prec^{\widetilde{\mathcal{E}}_1}_{\{b,c,a\}} \cap (\{c,a\} \times \{c,a\}) = \emptyset \neq \prec^{\widetilde{\mathcal{E}}_1 \setminus \{b\}}_{\{c,a\}} = \{(c,a)\}$, *although* $\{b\} \to \{b,c,a\}$ *in* $\widetilde{\mathcal{E}}_1$. *This is because* $\mathcal{E}_1$ *is not singular.*

Next, construct the RC-structure $(\widetilde{\mathcal{E}}_1 \setminus \{b\}) \setminus \{c\} = (\{a\}, \vdash^{**}_1, L, l^{**}_1 = l^*_1|_{\{a\}})$, *where* $\vdash^{**}{}_1 = \{(\emptyset,\emptyset)$, $(\emptyset,\{a\})$, $(\{a\},\emptyset)\}$.

At last, construct the RC-structure $\widetilde{\mathcal{E}}_1 \setminus (\{b\} \cup \{c\}) = (\{a\}, \vdash^{***}_1, L, l^{***}_1 = l^*_1|_{\{a\}})$, *where* $\vdash^{***}{}_1 = \{(\emptyset,\emptyset)$, $(\emptyset,\{a\})$, $(\{a\},\emptyset)\}$. *So, we see that* $\{b\} \to \{b,c\}$ *in* $\widetilde{\mathcal{E}}_1$, *and* $(\widetilde{\mathcal{E}}_1 \setminus \{b\}) \setminus \{c\} = \widetilde{\mathcal{E}}_1 \setminus (\{b\} \cup \{c\})$, *confirming the validity of Proposition 2(ii).*

Second, consider the rooted, pure, locally conjunctive and singular RC-structure $\mathcal{E}_5 = (E_5, \vdash_5, L, l_5)$ *from Example 1. Recall that* $E_5 = \{a,b,c\}$; $\vdash_5 = \{(\emptyset,\emptyset)$, $(\emptyset,\{a\})$, $(\emptyset,\{b\})$, $(\{a\},\{c\})$, $(\{b\},\{c\})$, $(\emptyset,\{a,c\})$, $(\emptyset,\{b,c\})\}$; $L = E_5$; *and* l_5 *is the identity labeling function. Moreover,* $LC(\mathcal{E}_5) = Conf(\mathcal{E}_5) = \{\emptyset,\{a\},\{b\},\{a,c\},\{b,c\}\}$, *and* $\prec^{\mathcal{E}_5}_{\{a,c\}} = \{(a,c)\}$ *and* $\prec^{\mathcal{E}_5}_{\{b,c\}} = \{(b,c)\}$.

From Example 2 we know that the enabling relation in the standard form $\widetilde{\mathcal{E}}_5$ *of* $\mathcal{E}_5$ *looks like this:* $\widetilde{\vdash}_5 = \{(\emptyset,\emptyset)$, $(\emptyset,\{a\})$, $(\{a\},\emptyset)$, $(\emptyset,\{b\})$, $(\{b\},\emptyset)$, $(\{a\},\{c\})$, $(\{c\},\ \{a\})$, $(\emptyset,\ \{a,c\})$, $(\{a,c\},\ \emptyset)$, $(\{b\},\{c\})$, $(\{c\},\ \{b\})$, $(\emptyset,\ \{b,c\})$, $(\{b,c\},\ \emptyset)\}$. *In addition,* $LC(\mathcal{E}_5) = LC(\widetilde{\mathcal{E}}_5)$, $Conf(\mathcal{E}_5) = Conf(\widetilde{\mathcal{E}}_5)$, *and* $\prec^{\widetilde{\mathcal{E}}_5}_{\{a,c\}} = \{(a,c)\}$, $\prec^{\widetilde{\mathcal{E}}_5}_{\{b,c\}} = \{(b,c)\}$. *Notice that* $\{b\} \to \{b,c\}$ *in* $\widetilde{\mathcal{E}}_5$.

Consider the RC-structure $\widetilde{\mathcal{E}}_5 \setminus \{b\} = (\{a,c\},\ \vdash^*_5,\ L,\ l^*_5)$, *where* $\vdash^*{}_5 = \{(\emptyset,\ \emptyset)$, $(\emptyset,\ \{c\})$, $(\{c\},\ \emptyset)\}$. *Then, we obtain* $Conf(\widetilde{\mathcal{E}}_5 \setminus \{b\}) = \{\emptyset,\ \{c\}\}$, *and* $\prec^{\widetilde{\mathcal{E}}_5 \setminus \{b\}}_{\{c\}} = \emptyset$. *Hence, it holds that* $\prec^{\widetilde{\mathcal{E}}_5}_{\{b,c\}} \cap (\{c\} \times \{c\}) = \prec^{\widetilde{\mathcal{E}}_5 \setminus \{b\}}_{\{c\}}$. *So, the example with* $\mathcal{E}_5$ *confirms the validity of Proposition 2(i(b)).*

4. Transition Systems $TC(\cdot)$ and $TR(\cdot)$ from RC-structures

In this section, we first give some basic definitions concerning labeled transition systems. Then, we define the mappings $TC(\mathcal{E})$ and $TR(\mathcal{E})$, which associate two distinct kinds of transition systems – one whose states are configurations and one whose states are residual RC-structures – with the RC-structure $\mathcal{E}$ labeled over the set L of labels.

A transition system $T = (S, \to, i)$ labeled over a set $\mathcal{L}$ of labels consists of a set of states S, a transition relation $\to \subseteq S \times \mathcal{L} \times S$, and an initial state $i \in S$. Two transition systems labeled over $\mathcal{L}$ are *isomorphic* if their states can be mapped one-to-one to each other, preserving transitions and initial states.

Let L be a fixed set of labels in RC-structures). Let $\mathbb{L}_{step} := \mathbb{N}_0^L$ (the set of multisets over L, or functions from L to the non-negative integers), and $\mathbb{L}_{pom} := Pom_L$ (the set of isomorphic classes of partial orders labeled over L) be another sets of labels. (The sets will be used as the set of labels in the transition systems.)

We are ready to define labeled transition systems with configurations as states.

Definition 4. *For a rooted RC-structure* $\mathcal{E}$ *over* L *and* $\star \in \{pom,\ step\}$,

$$TC_\star(\mathcal{E}) \text{ is a transition system } (Conf_\star(\mathcal{E}),\ \rightharpoondown_\star,\ \emptyset) \text{ over } \mathbb{L}_\star,$$

where $X \xrightharpoondown{p}_\star X'$ *iff* $X \to_\star X'$ *and* $p = l_\star(X' \setminus X)$ *in* $\mathcal{E}$.

We exemplify a feature of the above definition.

Example 4. *Consider the not rooted RC-structure* $\mathcal{E}_4 = (E_4, \vdash_4, L, l_4)$ *from Example 1. Recall that* $E_4 = \{a\}$; $\vdash_4 = \{(\emptyset, \{a\}), (\{a\}, \emptyset)\}$; $L = E_4$; *and* l_4 *is the identity labeling function. We know that* $LC(\mathcal{E}_4) = \{\{a\}\}$, *and* $Conf(\mathcal{E}_4) = \emptyset$, *as* $\mathcal{E}_4$ *is not rooted, i.e.* $(\emptyset, \emptyset) \not\in \vdash_4$. *We cannot construct the transition system* $TC_\star(\mathcal{E}_4) = (Conf_\star(\mathcal{E}_4),\ \rightharpoondown_\star,\ \emptyset)$ *over* $\mathbb{L}_\star$ *(*$\star \in \{pom,$ $step\}$*) because the initial state* $\emptyset$ *must belong to* $Conf(\mathcal{E}_4)$.

Lemma 4. *Given a rooted RC-structure* $\mathcal{E}$,

(i) $TC_{step}(\mathcal{E}) = TC_{step}(SF(\mathcal{E}))$, *if* $\mathcal{E}$ *is a pure RC-structure;*

(ii) $TC_{pom}(\mathcal{E}) = TC_{pom}(SF(\mathcal{E}))$, *if* $\mathcal{E}$ *is a pure, singular and locally conjunctive RC-structure.*

Proof. Let $\star \in \{step, pom\}$. Due to Lemma 2(i), we obtain that $Conf_\star(\mathcal{E}) = Conf(\mathcal{E})$ and $Conf_\star(SF(\mathcal{E})) = Conf(SF(\mathcal{E}))$. Since $\mathcal{E}$ is a pure RC-structure, we have that $Conf(\mathcal{E}) = Conf(SF(\mathcal{E}))$, by Proposition 1(ii). Thus, we obtain $Conf_\star(\mathcal{E}) = Conf_\star(SF(\mathcal{E}))$. Moreover, according to Lemma 2(ii), the relation $\to_\star$ in $\mathcal{E}$ is equal to the relation $\to_\star$ in $SF(\mathcal{E})$.

Thanks to the construction of $SF(\mathcal{E})$ and the definition of $l_{step}(\cdot)$, it easy to see that $l^{\mathcal{E}}_{step} = l^{SF(\mathcal{E})}_{step}$.

If $\mathcal{E}$ is a pure, singular and locally conjunctive RC-structure, it holds that $\preceq^{\mathcal{E}}_X = \preceq^{SF(\mathcal{E})}_X$, for all $X \in Conf(\mathcal{E}) = Conf(SF(\mathcal{E}))$, due to Proposition 1(iii). Hence, $l^{\mathcal{E}}_{pom} = l^{SF(\mathcal{E})}_{pom}$.

So, by Definition 4, we get $\rightharpoondown^{\mathcal{E}}_\star = \rightharpoondown^{SF(\mathcal{E})}_\star$. Thus, $TC_\star(\mathcal{E}) = TC_\star(SF(\mathcal{E}))$. □

We illustrate the validity of Lemma 4.

Example 5. *From Example 1 we know that* $\mathcal{E}_1$ *is a rooted and pure RC-structure,* $\mathcal{E}_3$ *is a rooted and not pure RC-structure, and* $\mathcal{E}_5$ *is a rooted, pure, singular and locally conjunctive*

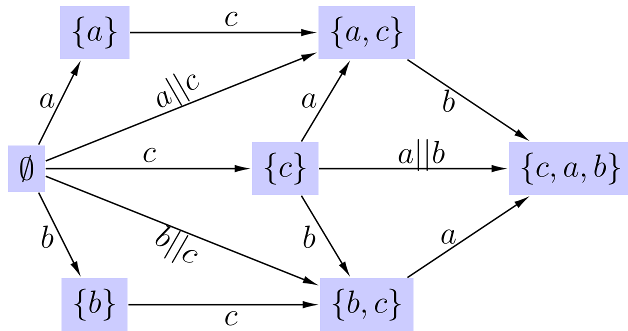

Fig. 1. The configuration transition system $TC_{step}(\mathcal{E}_1)$

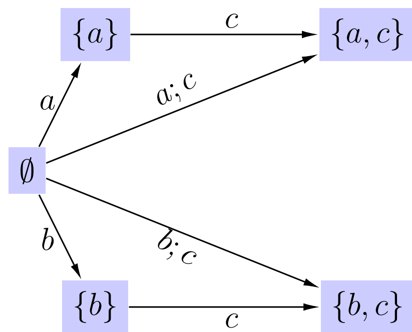

Fig. 2. The configuration transition system $TC_{pom}(\mathcal{E}_5)$

RC-structure. The transition system $TC_{step}(\mathcal{E}_1)$ is shown in Fig. 1, and the transition system $TC_{pom}(\mathcal{E}_5)$ is depicted in Fig. 2. Using Examples 3 and 2, respectively, it is easy to make sure that $TC_{step}(\mathcal{E}_1) = TC_{step}(SF(\mathcal{E}_1))$ and $TC_{pom}(\mathcal{E}_5) = TC_{pom}(SF(\mathcal{E}_5))$. However, this does not apply to $\mathcal{E}_3$ for both semantics. It is easy to see this by looking at Fig. 3 and 4.

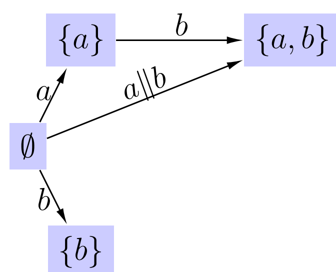

Fig. 3. The configuration transition system $TC_{step/pom}(\mathcal{E}_3)$

Next, introduce the definition of labeled transition systems with residuals as states.

Definition 5. *For an RC-structure $\mathcal{E}$ over L in standard form and $\star \in \{pom,\ step\}$,*

$$TR_\star(\mathcal{E}) \text{ is the transition system } (Reach_\star(\mathcal{E}),\ \rightharpoonup_\star,\ \mathcal{E}) \text{ over } \mathbb{L}_\star,$$

where $\mathcal{F} \overset{p}{\rightharpoonup}_\star \mathcal{F}'$ for some $p \in \mathbb{L}_\star$ iff $\mathcal{F}' = \mathcal{F} \setminus X$ and $\emptyset \rightarrow_\star X$ in $\mathcal{F}$, for some $X \in Conf_\star(\mathcal{F})$ with $p = l_\star(X \setminus \emptyset)$, and $Reach_\star(\mathcal{E}) = \{\mathcal{F} \mid \exists \mathcal{E}_0, \ldots, \mathcal{E}_k\ (k \geq 0)$ s.t. $\mathcal{E}_0 = \mathcal{E},\ \mathcal{E}_k = \mathcal{F}$, and

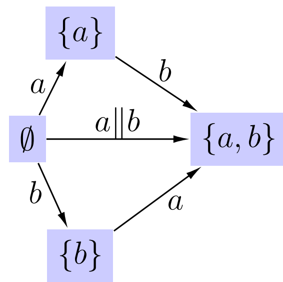

Fig. 4. The configuration transition system $TC_{step/pom}(SF(\mathcal{E}_3))$

$\mathcal{E}_i \xrightarrow{p}_\star \mathcal{E}_{i+1}$ $(i < k)\}$.

Example 6. *Consider* $\widetilde{\mathcal{E}}_1 = SF(\mathcal{E}_1)$ *in Example 3 and* $\widetilde{\mathcal{E}}_5 = SF(\mathcal{E}_5)$ *in Example 2. The transition system* $TR_{step}(\widetilde{\mathcal{E}}_1)$ *is shown in Fig. 5, and the transition system* $TR_{pom}(\widetilde{\mathcal{E}}_5)$ *is depicted in Fig. 6.*

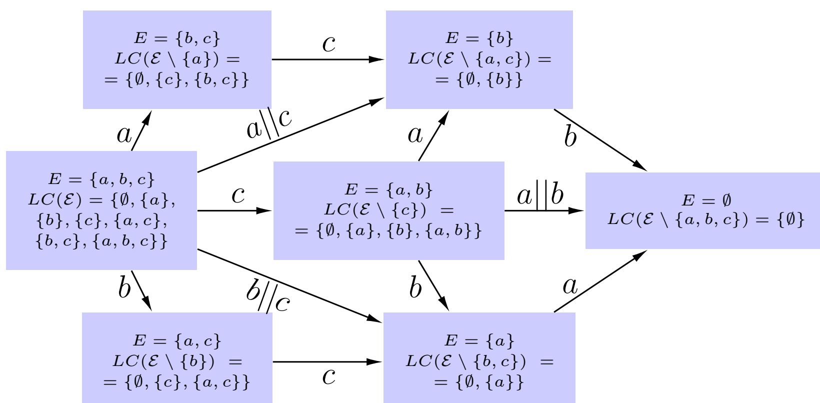

Fig. 5. The residual transition system $TR_{step}(SF(\mathcal{E}_1))$

We establish the relationships between states and transitions of $TC(\cdot)$ and $TR(\cdot)$ in both semantics.

Proposition 3. *Given a rooted RC-structure* $\mathcal{E}$ *in standard form and* $\star \in \{pom, step\}$,

(i) for any $X \in Conf_\star(\mathcal{E})$, $\mathcal{E} \setminus X \in Reach_\star(\mathcal{E})$;

(ii) for any $\mathcal{E}' \in Reach_\star(\mathcal{E})$, *there exists* $X \in Conf_\star(\mathcal{E})$ *such that* $\mathcal{E}' = \mathcal{E} \setminus X$;

(iii) for any $X', X'' \in Conf_\star(\mathcal{E})$, *if* $X' \xrightarrow{p}_\star X''$ *in* $TC_\star(\mathcal{E})$, *then* $\mathcal{E} \setminus X' \xrightarrow{p'}_\star \mathcal{E} \setminus X''$ *in* $TR_\star(\mathcal{E})$, *moreover,* $p = p'$ *if either* $\star = step$ *or* $\star = pom$ *and* $\mathcal{E}$ *is the standard form of a singular RC-structure;*

(iv) for any $\mathcal{E}', \mathcal{E}'' \in Reach_\star(\mathcal{E})$, *if* $\mathcal{E}' \xrightarrow{p}_\star \mathcal{E}''$ *in* $TR_\star(\mathcal{E})$, *then there are* $X', X'' \in Conf_\star(\mathcal{E})$ *such that* $\mathcal{E}' = \mathcal{E} \setminus X'$, $\mathcal{E}'' = \mathcal{E} \setminus X''$, *and* $X' \xrightarrow{p'}_\star X''$ *in* $TC_\star(\mathcal{E})$, *moreover,* $p = p'$ *if either* $\star = step$ *or* $\star = pom$ *and* $\mathcal{E}$ *is the standard form of a singular RC-structure.*

Proof.

(i) Take an arbitrary $X \in Conf_\star(\mathcal{E})$. Due to Lemma 2(i), we have that $Conf_\star(\mathcal{E}) = Conf(\mathcal{E})$. Hence, $X \in Conf(\mathcal{E})$. This implies that $X = \{e_1, \ldots, e_n\}$ $(n \geq 0)$ such that for all $i \leq n$ and for all $Y \subseteq \{e_1, \ldots, e_i\}$, there is $Z \subseteq \{e_1, \ldots, e_{i-1}\}$ such that $Z \vdash Y$. Clearly, $X_i = \{e_1, \ldots, e_i\} \in Conf(\mathcal{E})$ and $X_i \to X_{i+1}$, for all $i \leq n-1$. Let $Y_i = \{e_i\}$ for all $i \leq n$. Check that $\mathcal{E} \setminus X_i = \mathcal{E} \setminus Y_1 \setminus \ldots \setminus Y_i$, for all $i \leq n$. We shall proceed by induction on n.

$n = 0$. Obvious.

$n = 1$. Then, $Y_1 = X_1 \in Conf(\mathcal{E})$ and$\mathcal{E} \setminus X_1 = \mathcal{E} \setminus Y_1$.

$n > 1$. By the induction hypothesis, we have $\mathcal{E} \setminus X_{n-1} = \mathcal{E} \setminus Y_1 \setminus \ldots \setminus Y_{n-1}$. Since $X_{n-1}, X_n \in Conf(\mathcal{E})$ and $X_{n-1} \to X_n$ in $\mathcal{E}$, we obtain $X_n \setminus X_{n-1} = Y_n \in Conf(\mathcal{E} \setminus X_{n-1})$, due to Proposition 2(i(a)). According to Proposition 2(ii), it holds that $\mathcal{E} \setminus X_{n-1} \setminus Y_n = \mathcal{E} \setminus (X_{n-1} \cup Y_n) = \mathcal{E} \setminus X_n$. Hence, $\mathcal{E} \setminus Y_1 \setminus \ldots \setminus Y_{n-1} \setminus Y_n = \mathcal{E} \setminus X_n$.

It is easy to see that $\mathcal{E} \rightharpoonup_\star \mathcal{E} \setminus X_1 \rightharpoonup_\star \ldots \rightharpoonup_\star \mathcal{E} \setminus X_n$. Thus, $\mathcal{E} \setminus X \in Reach_\star(\mathcal{E})$.

(ii) Take an arbitrary $\mathcal{E}' \in Reach_\star(\mathcal{E})$. This means that $\mathcal{E} = \mathcal{E}_0 \overset{p_1}{\rightharpoonup}_\star \mathcal{E}_1 \ldots \mathcal{E}_{n-1} \overset{p_n}{\rightharpoonup}_\star \mathcal{E}_n = \mathcal{E}'$ $(n \geq 0)$. By the definition of $\overset{p_{i+1}}{\rightharpoonup}_\star$, it holds that $\mathcal{E}_{i+1} = \mathcal{E}_i \setminus X_{i+1}$, for some $X_{i+1} \in Conf_\star(\mathcal{E}_i)$ and $p_{i+1} = l_\star^{\mathcal{E}_i}(X_{i+1})$. Due to Lemma 2(i), we have $Conf_\star(\mathcal{E}_i) = Conf(\mathcal{E}_i)$. Hence, $X_{i+1} \in Conf(\mathcal{E}_i)$. Verify that $Y_{i+1} = \bigcup_{j=1}^{i+1} X_j \in Conf(\mathcal{E})$ and $\mathcal{E}_{i+1} = \mathcal{E} \setminus Y_{i+1}$, for all $i < n$. We shall proceed by induction on n.

$n = 0$. Obvious.

$n = 1$. Then, $Y_1 = X_1 \in Conf(\mathcal{E})$ and $\mathcal{E}_1 = \mathcal{E}_0 \setminus X_1 = \mathcal{E} \setminus Y_1$.

$n > 1$. By the induction hypothesis, $Y_{n-1} = \bigcup_{j=1}^{n-1} X_j \in Conf(\mathcal{E})$ and $\mathcal{E}_{n-1} = \mathcal{E} \setminus Y_{n-1}$. Check that $Y_n = \bigcup_{j=1}^{n} X_j \in Conf(\mathcal{E})$ and $\mathcal{E}_n = \mathcal{E} \setminus Y_n$. As $\mathcal{E}_n = \mathcal{E}_{n-1} \setminus X_n$, it holds that $\mathcal{E}_n = (\mathcal{E} \setminus Y_{n-1}) \setminus X_n$. According to Proposition 2(ii), we have that $Y_{n-1} \cup X_n = Y_n \in Conf(\mathcal{E})$ and $\mathcal{E}_n = \mathcal{E} \setminus (Y_{n-1} \cup X_n) = \mathcal{E} \setminus Y_n$. Thus, $\mathcal{E}' = \mathcal{E} \setminus Y_n$. Moreover, it is true that $Y_n \in Conf_\star(\mathcal{E})$, due to Lemma 2(i).

(iii) Take arbitrary $X', X'' \in Conf_\star(\mathcal{E})$ such that $X' \overset{p}{\to}_\star X''$ in $TC_\star(\mathcal{E})$. Then, we have that $X' \to_\star X''$ in $\mathcal{E}$ and $p = l_\star^{\mathcal{E}}(X'' \setminus X')$. This means that $X' \to X''$. As $\mathcal{E}$ is an RC-structure in standard form, we obtain that $X'' \setminus X' \in Conf(\mathcal{E} \setminus X')$, by Propositions 2(i(a)). Moreover, since $X' \to_\star X''$ in $\mathcal{E}$, we may conclude that $\emptyset \to_\star X'' \setminus X'$ in $\mathcal{E} \setminus X'$, by the construction of $\mathcal{E} \setminus X'$. Next, due to Propositions 2(ii), it holds that $\mathcal{E} \setminus X'' = (\mathcal{E} \setminus X') \setminus (X'' \setminus X')$. Then, $\mathcal{E} \setminus X' \overset{p'}{\rightharpoonup}_\star \mathcal{E} \setminus X''$ in $TR_\star(\mathcal{E})$, where $p' = l_\star^{\mathcal{E} \setminus X'}(X'' \setminus X')$. Moreover, if $\star = step$, we get that $p = l_{step}^{\mathcal{E}}(X'' \setminus X') = l_{step}^{\mathcal{E} \setminus X'}(X'' \setminus X') = p'$. If $\star = pom$ and $\mathcal{E}$ is the standard form of a singular RC-structure, we have $\prec^{\mathcal{E}\setminus X'}_{X''\setminus X'} = \prec^{\mathcal{E}}_{X''} \cap (X''\setminus X' \times X''\setminus X')$, by Proposition 2(i(b)). Hence, $p = l^{\mathcal{E}}_{pom}(X'' \setminus X') = l^{\mathcal{E}\setminus X'}_{pom}(X'' \setminus X') = p'$.

(iv) Take arbitrary $\mathcal{E}', \mathcal{E}'' \in Reach_\star(\mathcal{E})$ such that $\mathcal{E}' \xrightarrow{p}_\star \mathcal{E}''$ in $TR_\star(\mathcal{E})$. Due to item (ii), there is $X' \in Conf_\star(\mathcal{E})$ such that $\mathcal{E}' = \mathcal{E} \setminus X'$. According to the definition of $\xrightarrow{p}_\star$, there is $\widetilde{X}' \in Conf_\star(\mathcal{E}')$ such that $\mathcal{E}'' = \mathcal{E}' \setminus \widetilde{X}'$, $\emptyset \to_\star \widetilde{X}'$ in $\mathcal{E}'$ and $p = l^{\mathcal{E}'}_\star(\widetilde{X}')$. Then, $X'' = X' \cup \widetilde{X}' \in Conf_\star(\mathcal{E})$, $X' \to X''$, and $\mathcal{E}'' = \mathcal{E} \setminus X''$, due to Proposition 2(ii). As $\emptyset \to_\star \widetilde{X}'$ in $\mathcal{E}'$ and $X' \to X''$, we have that $X' \xrightarrow{p'}_\star X''$ in $TC_\star(\mathcal{E})$, where $p' = l^{\mathcal{E}}_\star(X'' \setminus X')$. Moreover, if $\star = step$, we get that $p' = l^{\mathcal{E}}_{step}(X'' \setminus X') = l^{\mathcal{E}\setminus X'}_{step}(X'' \setminus X') = p$. If $\star = pom$ and $\mathcal{E}$ is the standard form of a singular RC-structure, it holds that $\prec^{\mathcal{E}\setminus X'}_{X''\setminus X'} = \prec^{\mathcal{E}}_{X''} \cap (X''\setminus X' \times X''\setminus X')$, by Proposition 2(i(b)). Hence, $p' = l^{\mathcal{E}}_{pom}(X'' \setminus X') = l^{\mathcal{E}\setminus X'}_{pom}(X'' \setminus X') = l^{\mathcal{E}'}_{pom}(\widetilde{X}') = p$. □

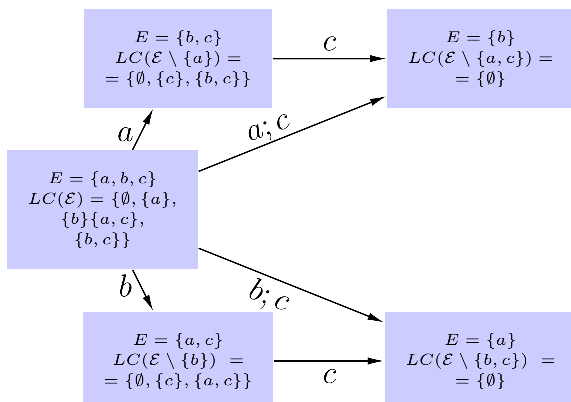

Fig. 6. The residual transition system $TR_{pom}(SF(\mathcal{E}_5))$

Finally, we formulate the main results of the paper.

Theorem 1. *Given a rooted RC-structure* $\mathcal{E}$ *and* $\star \in \{step, pom\}$,

(i) $TC_{step}(\mathcal{E})$ *and* $TR_{step}(SF(\mathcal{E}))$ *are isomorphic, if* $\mathcal{E}$ *is a pure RC-structure;*

(ii) $TC_{pom}(\mathcal{E})$ *and* $TR_{pom}(SF(\mathcal{E}))$ *are isomorphic, if* $\mathcal{E}$ *is a pure, singular and locally conjunctive RC-structure.*

Proof. Let $\star \in \{step, pom\}$. First, define a mapping $g : Conf_\star(SF(\mathcal{E})) \to Reach_\star(SF(\mathcal{E}))$ as follows: $g(X) = SF(\mathcal{E}) \setminus X$ for all $X \in Conf_\star(SF(\mathcal{E}))$. Since $\mathcal{E}$ is a rooted RC-structure, it holds that $SF(\mathcal{E})$ is a rooted RC-structure, by Proposition 1(i). Then, $g(X)$ is well-defined, due to Proposition 3(i).

Since $SF(\mathcal{E})$ is a rooted RC-structure, it is clear that $\emptyset \in Conf(SF(\mathcal{E}))$. Thanks to Lemma 2(i), we get that $Conf_\star(SF(\mathcal{E})) = Conf(SF(\mathcal{E}))$. Hence, $g(\emptyset) = SF(\mathcal{E}) \setminus \emptyset = SF(\mathcal{E})$.

Check that g is a bijective mapping. Assume $g(X) = g(X')$, for some $X, X' \in Conf_\star(SF(\mathcal{E}))$.

This means that $SF(\mathcal{E}) \setminus X = SF(\mathcal{E}) \setminus X'$. By Definition 3 and the construction of $SF(\mathcal{E})$, we get $E \setminus X = E \setminus X'$. Since $X, X' \subseteq E$, we have $X = X'$. So, g is an injective mapping.

Take an arbitrary $\mathcal{E}' \in Reach_\star(SF(\mathcal{E}))$. Due to Proposition 3(ii), we get $\mathcal{E}' = SF(\mathcal{E}) \setminus X$, for some $X \in Conf_\star(SF(\mathcal{E}))$. Hence, g is a surjective mapping.

(i) According to Propositions 3(iii) and (iv), and the fact that g is a bijective mapping, we have that $X \xrightarrow{p}_{step} X'$ in $TC_{step}(SF(\mathcal{E}))$ iff $g(X) \xrightarrow{p}_{step} g(X')$ in $TR_{step}(SF(\mathcal{E}))$. So, g is indeed an isomorphism between $TC_{step}(SF(\mathcal{E}))$ and $TR_{step}(SF(\mathcal{E}))$. Due to Lemma 4 (i), we may conclude that $TC_{step}(\mathcal{E}) = TC_{step}(SF(\mathcal{E}))$, because $\mathcal{E}$ is a pure RC-structure. Thus, $TC_{step}(\mathcal{E})$ and $TR_{step}(SF(\mathcal{E}))$ are isomorphic.

(ii) Since $SF(\mathcal{E})$ is the standard form of a singular RC-structure and g is a bijective mapping, we obtain that $X \xrightarrow{p}_{pom} X'$ in $TC_{pom}(SF(\mathcal{E}))$ iff $g(X) \xrightarrow{p}_{pom} g(X')$ in $TR_{pom}(SF(\mathcal{E}))$, by Propositions 3(iii) and (iv). So, g is indeed an isomorphism. Since $\mathcal{E}$ is a pure, singular and locally conjunctive RC-structure, we get that $TC_{pom}(\mathcal{E}) = TC_{pom}(SF(\mathcal{E}))$, by Lemma 4 (ii). Thus, $TC_{pom}(\mathcal{E})$ and $TR_{pom}(SF(\mathcal{E}))$ are isomorphic. □

Example 7. *From example 1 we know that $\mathcal{E}_1$ is a rooted, pure, not singular, and locally conjunctive RC-structure, and $\mathcal{E}_5$ is a rooted, pure, singular and locally conjunctive RC-structure. It is easy to see that $TC_{step}(\mathcal{E}_1)$ shown in Fig. 1 and $TR_{step}(SF(\mathcal{E}_1))$ shown in Fig. 5 are isomorphic, and the transition system $TC_{pom}(\mathcal{E}_5)$ depicted in Fig. 2 and $TR_{step}(SF(\mathcal{E}_5))$ shown in Fig. 6 are isomorphic. However, this does not apply to $\mathcal{E}_1$ in partial order semantics because the structure is not singular. In Fig. 7 and 8, we see that $\emptyset \overset{a\|c}{\to} \{a, c\}$ and $\emptyset \overset{b\|c}{\to} \{b, c\}$ in $TC_{pom}(\mathcal{E}_1)$, and $\mathcal{E}_1 \overset{c;a}{\to} \mathcal{E}_1 \setminus \{a, c\}$ and $\emptyset \overset{c;b}{\to} \mathcal{E}_1 \setminus \{b, c\}$ in $TR_{pom}(SF(\mathcal{E}_1))$.*

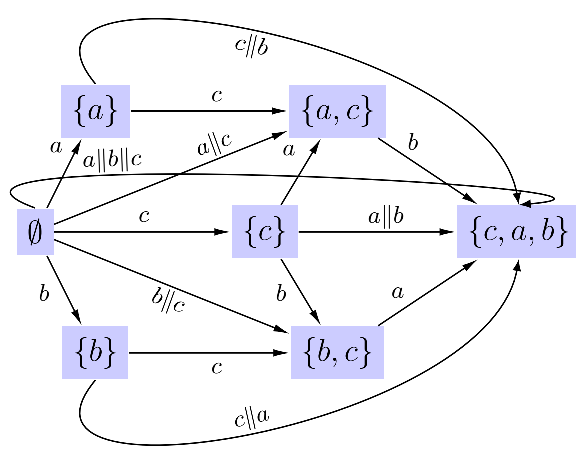

Fig. 7. The configuration transition system $TC_{pom}(\mathcal{E}_1)$

5. Concluding Remarks

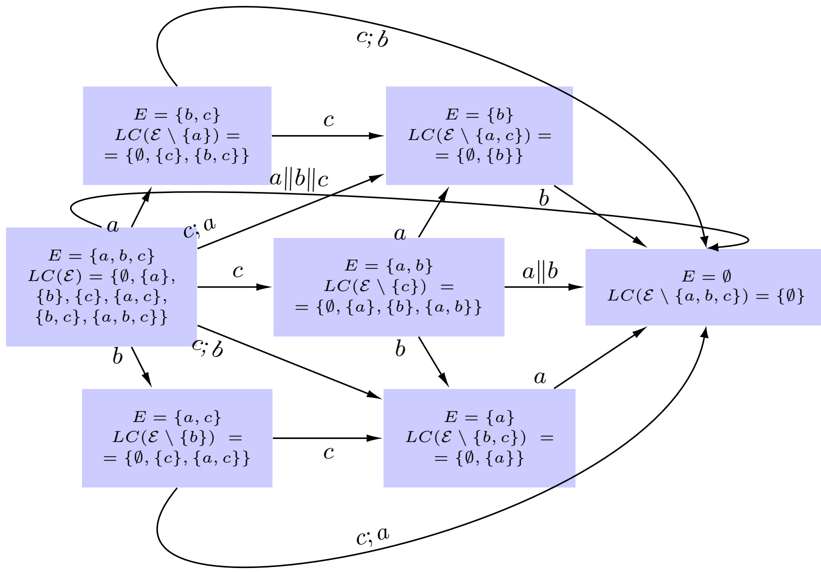

Fig. 8. The residual transition system $TR_{pom}(SF(\mathcal{E}_1))$

In this paper, we investigated two different ways of providing various transition system semantics for RC-structures which are very expressive among the event-oriented models known in the literature. First, we have developed the notion of partial orders within the configurations of RC-structures and discovered their subclasses in which partial orders are preserved in the standard form of the models and under the removal operator. Second, we have revealed closed relationships between configurations in the original RC-structure, in its standard form and in its residuals, under some conditions. Third, we have formulated properties of the model under consideration, which guarantee the coincidence of configuration transition systems obtained from the RC-structure and its standard form. As our main result, we have demonstrated how transition systems based on configurations and residuals of RC-structures, presented not necessary in standard form, are related in the context of partial order multiset and step semantics.

Work is currently underway to extend our approach to other event-oriented models (e.g., to precursor [11], probabilistic [28], and local [18] event structures, and also to event structures with dynamic causality [1]), and it gave preliminary results. Another future direction of our research is to extend our results in comparing of the two types of transition systems studied to the multiset case of transition relation and to the non-pure case of RC-structures.

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